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THE GENERAL EULERIAN INTEGRAL WITH ALEPH (ℵ)-FUNCTION

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Abstract: The aim of the Present paper is to evaluating the general class of Eulerian integrals involving general class of polynomials and Aleph (ℵ)-function. Our main result (22) below is shown to provide key formula from which many integrals can be deduced.

Keywords: Aleph (ℵ)-function, Beta Integral, general class of polynomials, I-function.

Mathematics Subject Classification: 33C60, 11S23, 33C45.

1. Introduction

For our Purpose we begin by recalling some known functions and earlier works, by the definition of Gamma and Beta functions, it is known that the Eulerian beta integral

$$B\left(\alpha,\beta\right)=\int_{0}^{1}t^{\alpha-1}\left(1-t\right)^{\beta-1}dt=\frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)};\left[Re\left(\alpha\right)>0;Re\left(\beta\right)>0\right] \tag{1}$$

Can be again in its presented as

$$\int_{a}^{b}\left(t-a\right)^{\alpha-1}\left(b-t\right)^{\beta-1}dt=\left(b-a\right)^{\alpha+\beta-1}B\left(\alpha,\beta\right);\left[Re\left(\alpha\right)>0;Re\left(\beta\right)>0;a\neq b\right] \tag{2}$$

Since

$$(ut+\nu)^{\gamma}=(au+\nu)^{\gamma}\times$$

$$\sum_{l=0}^{\infty}\frac{(-\gamma)_l}{l!}\left\{-\frac{(t-a)\,u}{au+\nu}\right\}^{l};[|\,(t-a)\,u|<|\,(au+\nu)\,|;t\in[a,b]] \tag{3}$$

We find from (2) that [2, p.301, eq. 2.2.6.1]

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+\nu)^{\gamma}\,dt$$

$$=(b-a)^{\alpha+\beta-1}(au+\nu)^{\gamma}B(\alpha,\beta)\ \ {}_2F_1\left[\alpha,-\gamma;\alpha+\beta;-\frac{(b-a)\,u}{au+\nu}\right] \tag{4}$$

$$\left[Re\,(\alpha)>0;Re\,(\beta)>0;|arg\left(\frac{bu+\nu}{au+\nu}\right)|\leq\pi-\varepsilon\,(0<\varepsilon<\pi)\,;a\neq b\right]$$

Putting $\gamma=-\alpha-\beta$ in (4) would simplify considerably, and if further we Put $u=\lambda-\mu$ and $\nu=(1+\mu)\,b-(1+\lambda)\,a$ in terms of new parameters λ and μ, the special case $\gamma=-\alpha-\beta$ of (4) would yield (Gradshteyn, 1980[8]; p. 287, eq. 3.198); (Prudnikov et at, 1983 [2, p, 301, eq. 2.2.6.1])

$$\int\limits_a^b\frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda\,(t-a)+\mu\,(b-t)\}^{\alpha+\beta}}dt=\frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{(b-a)}B\,(\alpha,\beta) \tag{5}$$

$[Re\,(\alpha)>0;Re\,(\beta)>0;b-a+\lambda\,(t-a)+\mu\,(b-t)\neq 0,t\epsilon\,[a,b]\,;a\neq b]$

By Use of (5) we show the problem of closed form evaluation of the following general Eulerian Integral with Aleph ($\aleph$)-function. Which is as follows

$$\Re=\int_l^w\frac{(t-l)^{\lambda}(w-t)^{\mu}}{\{f\,(t)\}^{\lambda+\mu+2}}\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\,\{g\,(t)\}^{\nu}\,|^{(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}}\right]\times$$

$$S^U_V\left[y\,\{g\,(t)\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\,\{g\,(t)\}^{\varsigma}\right]dt \tag{6}$$

where

$$f\,(t)=w-l+\rho\,(t-l)+\sigma\,(w-l) \tag{7}$$

$$g\,(t)=\frac{(t-l)^{\gamma}(w-t)^{\delta}\{f\,(t)\}^{1-\gamma-\delta}}{\beta\,(w-l)+(\beta\rho+\alpha-\beta)\,(t-l)+\beta\sigma\,(w-t)} \tag{8}$$

The Aleph (ℵ) -function, introduced by Sudland [4], here the notation and complete definition is presented in the following manner which is given below in terms on the Mellin- Barnes type integrals.

$$\aleph\left[z\right]=\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\Big|^{(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}}\right]$$

$$=\frac{1}{2\pi\omega}\int_L\Omega^{M,N}_{P_i,Q_i,\tau_i,r}\left(s\right)z^{-s}ds \tag{9}$$

For all z$\neq$ 0 where $\omega=\surd\left(-1\right)$

$$\Omega^{M,N}_{P_i,Q_i,\tau_i,r}\left(s\right)=\frac{\prod_{j=1}^{M}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{N}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=M+1}^{Q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod_{j=N+1}^{P_i}\Gamma\left(a_{ji}+A_{ji}s\right)} \tag{10}$$

The integration path $L=L_{i\gamma\infty},\gamma\epsilon R$ extends from $\gamma-i\infty$ to $\gamma+i\infty$, and is such that the poles, assumed to be simple of $\Gamma\left(1-\alpha_i-A_js\right),\ j=1,...,n$ do not coincide with the pole of $\Gamma\left(\beta_i+B_js\right),\ j=1,...,m$ the parameter $x_i,\ y_i$ are non negative integers satisfying $0\leq N\leq P_i,\ 0\leq M\leq Q_i,\ \tau_i>0$ for $i=1,...,r$. The $A_j,\ B_j,\ A_{ji},\ B_{ji}>0$ and $a_j,\ b_j,\ a_{ji},\ b_{ji}\epsilon C$. The empty product in (10) is interpreted as unity. The existence conditions for the defining integral (9) are giving below

$$\phi_l>0,\left|arg\left(z\right)\right|<\frac{\pi}{2}\phi_l,l=1,...,r \tag{11}$$

$$\phi_l\geq 0,\ \left|arg\left(z\right)\right|<\frac{\pi}{2}\phi_l\text{ and }R\left(\xi_l\right)+1<0 \tag{12}$$

where

$$\phi_l=\sum_{j=1}^{N}A_j+\sum_{j=1}^{M}B_j-\tau_l\left(\sum_{j=N+1}^{P_l}A_{jl}+\sum_{j=M+1}^{Q_l}B_{jl}\right) \tag{13}$$

$$\xi_l=\sum_{j=1}^{M}b_j-\sum_{j=1}^{N}a_j+\tau_l\left(\sum_{j=M+1}^{Q_l}b_{jl}-\sum_{j=N+1}^{P_l}a_{jl}\right)+\frac{1}{2}\left(P_l-Q_l\right),\left(i-1,...,r\right) \tag{14}$$

The Srivastava general class of Polynomials introduced by Srivastava defined as follows [6].

$$S^U_V\left[x\right]=\sum_{R=0}^{[V/U]}\left(-V\right)_{UR}A\left(V,R\right)\frac{x^R}{R!} \tag{15}$$

where U is an arbitrary positive integers, V=0, 1, 2, ..., r and the coefficients A(V, R) are arbitrary constant, real or complex. A number of well known polynomials follow as special cases of S_V^U as referred is [1].

The generalized polynomial set is defined by the following Rodrigues type formula [10, p.64, eq. (2.3.4)].

$$S_n^{\mu,\delta,\tau}\left[x; w, s, q, A, B, m, \xi, l\right]$$

$$= (Ax+B)^{-\mu}\left(1-\tau x^w\right)^{-\delta/\tau} T_{\xi,l}^{m+n}\left[(Ax+B)^{\mu+qn}\left(1-\tau x^w\right)^{\left(\frac{\delta}{\tau}\right)+sn}\right] \tag{16}$$

With the differential operator

$T_{k,l} = x^l\left[k + x\dfrac{d}{dx}\right]$

The explicit series form this generalized sequence of functions is given by [11, p.64, eq. (2.3.4)]

$$S_n^{\mu,\delta,\tau}\left[x; w, s, q, A, B, m, \xi, l\right] =$$

$$B^{qn}x^{l(m+n)}\left(1-\tau x^l\right)^{sn} l^{m+n}\sum_{\sigma=0}^{m+n}\sum_{\tau=0}^{\sigma}\sum_{j=0}^{m+n}\sum_{i=0}^{j}\frac{(-1)^j(-j)_i(\mu)_i(-\sigma)_i(-\mu-qn)_i}{\sigma!\tau!i!(1-\mu-j)_i}$$

$$\times\left(-\frac{\delta}{\tau}-sn\right)_\sigma\left(\frac{i+\xi+w\tau}{l}\right)_{m+n}\left(\frac{-\tau x^w}{1-\tau x^w}\right)\left(\frac{Ax}{B}\right)^i \tag{17}$$

Some special cases of (17) are given by Raijada in table form [11]. We shall use the following special case:

If we Substitute A=1, B=0 in (17) and letting $\tau \to 0$ and using the well known results.

$$Lt_{\tau\to 0}\left(1-\tau x^w\right)^{\delta/\tau} = exp\left(-\delta x^\tau\right),\ Lt_{|b|\to\infty}(b)_n\left(\frac{z}{b}\right)^n = z^n$$

then, we arrive at the following important polynomial set

$$S_n^{\mu,\delta,0}[x] = S_n^{\mu,\delta,0}\left[x; w, s, q, 1, 0, m, \xi, l\right]$$

$$= x^{qn+l(m+n)}l^{m+n}\sum_{\sigma=0}^{m+n}\sum_{\tau=0}^{\sigma}\frac{(-\sigma)_\tau\left(\frac{\mu+qn+\xi+w\tau}{l}\right)_{m+n}\left(\delta x^w\right)^\sigma}{\sigma!\tau!} \tag{18}$$

2. Main Integral

Evaluation of our main integral representing the general Eulering integral (6) making use of definition (9), (15) and (18), we find from (6) that

$$\Re = \int_l^w \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^{\lambda+\mu+2}} \aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\{g(t)\}^\nu \Big|_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}}^{(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}\right] \times$$

$$S_V^U\left[y\{g(t)\}^{\rho'}\right] S_n^{\alpha',\beta',0}\left[x\{g(t)\}^{\varsigma}\right] dt$$

$$= \sum_{R=0}^{V/U} \frac{(-V)_{UR} A_{V,R}}{R!} (y)^R (x)^{R'} (h)^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} \frac{(-\eta)_e \left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n} (\beta' x^\tau)^\eta}{\eta! e!} \times$$

$$\left[\int_l^w \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^{\lambda+\mu+2}} \times \frac{1}{2\pi i}\int_L \Omega_{P_i,Q_i,\tau_i,r}^{M,N}(s)\, z^{-s} \{g(t)\}^{-\nu s+\rho' R+\varsigma R'+\tau\eta} ds\right] dt \quad (19)$$

Where $R' = qn + h(m+n)$, $\Lambda = \rho' R + \varsigma R' + \tau\eta$ and L is suitable contour of Mellin Barnes type in the complexes plane $f(t)$, $g(t)$, $\Omega(s)$ are given by (7), (8) and (10) respectively. Assuming the inversion of the order of integration in (19) be provided by absolute (and uniform) convergence of the involved above, we have

$$\Re = \sum_{R=0}^{V/U} \frac{(-V)_{UR} A_{V,R}}{R!} (y)^R (x)^{R'} (h)^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} \frac{(-\eta)_e \left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n} (\beta' x^\tau)^\eta}{\eta! e!} \times$$

$$\left[\frac{1}{2\pi i}\int_L \Omega_{P_i,Q_i,\tau_i,r}^{M,N}(s) \left(\frac{z}{\beta^\nu}\right)^{-s} \frac{1}{\beta^\Lambda}\right] \times$$

$$\left[\int_l^w \frac{(t-l)^{\lambda-\nu\gamma s+\Lambda\gamma} (w-t)^{\mu-\delta\nu s+\Lambda s}}{\{f(t)\}^{\lambda+\mu+(\gamma+\delta)(-\gamma s+\Lambda)+2}} \left\{1 - \frac{(\beta-\alpha)(t-l)}{\beta f(t)}\right\}^{\nu s-\Lambda} ds\right] dt \qquad (20)$$

If $|(\beta-\alpha)(t-l)| < |\beta f(t)|$, $(t \epsilon [l, w])$ then we can be made of binomial expansion and we thus find from (20) that

$$\Re = \sum_{R=0}^{V/U} \frac{(-V)_{UR} A_{V,R}}{R!} \left(\frac{y}{\beta^{\rho'}}\right)^R \left(\frac{x}{\beta^\varsigma}\right)^{R'} (h)^{m+n} \times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta' x^{\tau}\right)^{\eta}}{\eta!e!}\times$$

$$\left[\frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i,r}(s)\left(\frac{z}{\beta^{\nu}}\right)^{-s}\int_l^w \frac{(t-l)^{(\lambda+k-\nu\gamma s+\Lambda\gamma+1)-1}(w-t)^{(\mu-\delta\nu s+\Lambda s+1)-1}}{\{f(t)\}^{\lambda+\mu+k+(\gamma+\delta)(-\gamma s+\Lambda)+2}}\right]\times$$

$$\left[\sum_{k=0}^{\infty}\left(\frac{\beta-\alpha}{\beta}\right)^{k}\frac{\Gamma(-\nu s+\Lambda+k)}{\Gamma(-\nu s+\Lambda)(k)!}dt\right]ds \tag{21}$$

Provided also that the order of summation and integration can be inverted. The inner most integral in (21) can be evaluated by appearing to known extension of the Eulerian (beta-function) integral (5) and finally obtain the desired integral formula:

$$\Re=(w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\Lambda\delta-1}\times$$

$$\sum_{R=0}^{V/U}\frac{(-V)_{UR}A_{V,R}}{R!}\left(\frac{y}{\beta\rho'}\right)^{R}\left(\frac{x}{\beta\varsigma}\right)^{R'}(h)^{m+n}\times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta'\left(\frac{x}{\beta}\right)^{\tau}\right)^{\eta}}{\eta!e!}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\beta-\alpha}{\beta(1+\rho)}\right)^{k}\aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}$$

$$\left[z\left\{\frac{\beta(1+\rho)^{\gamma}}{(1+\sigma)^{-\delta}}\right\}^{-\nu}\Big|^{(1-k-\Lambda,\nu),(-k-\lambda-\gamma\Lambda,\gamma\nu),(-\mu-\delta\Lambda,\delta\nu),(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i},(1-\Lambda,\nu),\{-k-\lambda-\mu-(\gamma+\delta)\Lambda-1,\nu(\gamma+\delta)\}}\right] \tag{22}$$

Which hold true when

i $V>0;\ \gamma>0;\ \delta>0;\ \beta\neq 0;\ w\neq l;\ \rho,\ \sigma\neq -1$ and $\{w-l+\rho(t-l)+\sigma(w-t)\}\neq 0,\ t\epsilon[l,\ w]$

ii $Re\left\{1+\lambda+\gamma\nu\left(\frac{b_j}{\beta_j}\right)\right\}>0$ and $Re\left\{1+\mu+\nu\delta\left(\frac{b_j}{a_j}\right)\right\}>0,\ \ (j=1,\ \ldots,\ M)$ where M is an arbitrary positive integer.

iii M, N, P, Q are positive integers constrainted by $1\leq M\leq Q,\ 0\leq N\leq P$.

iv $|arg(z)|<\frac{1}{2\pi\Omega}$.

v $|(\beta-\alpha)(t-l)|<|\beta\{w-l+\rho(t-l)+\sigma(w-l)\}|,\ t\epsilon[a,\ b]$.

vi U is an arbitrary positive integer and the $A_{V,\ R}\,(V,\ R=0)$ coefficients are arbitrary constants real or Complex.

vii The series on the right hand side of (22) converges absolutely.

3. Application

In this section we specifically show how the general integral formula (22) can be applied to derive various interesting results including those given by [9].
For all $\rho=\sigma=0$ and $z=(w-l)^{(\gamma+\delta-1)\nu}$ in (22) reduces

$$\Re=(w-l)^{-1}\sum_{R=0}^{V/U}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\varsigma}}\right)^{R'}(h)^{m+n}\times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta'\left(\frac{x}{\beta}\right)^{\tau}\right)^{\eta}}{\eta!e!}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\beta-\alpha}{\beta}\right)^{k}\aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}$$

$$\left[\left\{\frac{(w-l)^{\gamma+\delta-1}}{\beta}\right\}^{-\nu}\Big|^{(1-k-\Lambda,\nu),(-k-\lambda-\gamma\Lambda,\gamma\nu),(-\mu-\delta\Lambda,\delta\nu),(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i},(1-\Lambda,\nu),\{-k-\lambda-\mu-(\gamma+\delta)\Lambda-1,\nu(\gamma+\delta)\}}\right] \tag{23}$$

Provided that the conditions easily obtainable from those of (22) are satisfied. Taking $\beta=\alpha=\dfrac{1}{\varepsilon}$ in (23) we obtain

$$\Re=(w-l)^{-1}\sum_{R=0}^{V/U}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left(y\varepsilon^{\rho'}\right)^{R}(x\varepsilon^{\varsigma})^{R'}(h)^{m+n}\times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta'(x\varepsilon)^{\tau}\right)^{\eta}}{\eta!e!}\aleph^{M,N+3}_{P_i+3,Q_i+2,\tau_i;r}$$

$$\left[\left\{\varepsilon(w-l)^{\gamma+\delta-1}\right\}^{-\nu}\Big|^{(1-k-\Lambda,\nu),(-k-\lambda-\gamma\Lambda,\gamma\nu),(-\mu-\delta\Lambda,\delta\nu),(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i},(1-\Lambda,\nu),\{-k-\lambda-\mu-(\gamma+\delta)\Lambda-1,\nu(\gamma+\delta)\}}\right] \tag{24}$$

Again we put $\gamma=\delta=1$, $\lambda=\mu=-\frac{1}{2}$, $\alpha\to\alpha^2$ and $\beta\to\beta^2$ in the integral formula (23), and sum the resulting series by mean of a known formula [3]: applying Legendre's Duplication formula as well,we obtain the integral

$$\int_l^w(t-l)^{-1/2}(w-t)^{-1/2}$$

$$\aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\left\{\frac{(t-l)(w-t)}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\nu}\Big|_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}}^{(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}\right]\times$$

$$S_V^U\left[y\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\left\{\frac{(t-l)(w-t)(w-l)^{-1}}{\alpha^2(t-l)+\beta^2(w-t)}\right\}^{\varsigma}\right]dt$$

$$=(\sqrt{\pi})\sum_{R=0}^{V/U}\frac{(-V)_{UR}A_{V,R}}{R!}\left(\frac{y}{\beta^{2\rho'}}\right)^{R}\left(\frac{x}{\beta^{2\varsigma}}\right)^{R'}(h)^{m+n}\times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta'\left(\frac{x}{\beta^2}\right)^{\tau}\right)^{\eta}}{\eta!e!}\times$$

$$\aleph_{P_i+1,Q_i+1,\tau_i;r}^{M,N+1}\left[z\left\{\frac{(w-l)^{\gamma}}{(\alpha+\beta)^2}\right\}^{-\nu}\Big|_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i},(-\Lambda,\nu)}^{\left(\frac{1}{2}-\Lambda,\nu\right),(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}\right]\tag{25}$$

If we set $\gamma=\delta=\frac{1}{2}$, $\mu=-\lambda-2$ and $V\to 2V$ in the integral formula (23), and sum the resulting series by mean of a known formula [3]: applying Legendre's Duplication formula as well, we obtain the integral

$$\int_l^w (t-l)^{\lambda}(w-t)^{-\lambda-2}$$

$$\aleph_{P_i,Q_i,\tau_i,r}^{M,N}\left[z\left[\frac{\{(t-l)(w-t)\}^{\nu}}{\{\alpha(t-l)+\beta(w-t)\}^{2\nu}}\right]\Big|_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}}^{(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}\right]\times$$

$$S_V^U\left[y\left\{\frac{\left\{(t-l)^{1/2}(w-t)\right\}^{1/2}}{\{\alpha(t-l)+\beta(w-t)}\right\}^{\rho'}\right]S_n^{\alpha',\beta',0}\left[x\left\{\frac{\left\{(t-l)^{1/2}(w-t)\right\}^{1/2}}{\{\alpha(t-l)+\beta(w-t)}\right\}^{\varsigma}\right]dt$$

$$=(\sqrt{\pi})\sum_{R=0}^{V/U}\frac{(-V)_{UR}A_{V,R}}{R!}2^{1-\Lambda}\left(\frac{y}{\beta^{\rho'}}\right)^{R}\left(\frac{x}{\beta^{\varsigma}}\right)^{R'}\left(\frac{\beta}{\alpha}\right)^{1+\lambda+\frac{1}{2}-\Lambda}(h)^{m+n}\times$$

$$\sum_{\eta=0}^{m+n}\sum_{e=0}^{\eta}\frac{(-\eta)_e\left(\frac{\alpha'+qn+\xi+\tau e}{h}\right)_{m+n}\left(\beta'\left(\frac{x}{\beta^{\varsigma}}\right)^{\tau}\right)^{\eta}}{\eta!e!}\times$$

$$\aleph_{P_i+2,Q_i+2,\tau_i;r}^{M,N+2}\left[\{4\alpha\beta\}^{-\nu}\Big|_{(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i},\left(1-\frac{1}{2}-\Lambda,\nu\right),\left(\frac{1}{2}-\frac{1}{2}\Lambda,\nu\right)}^{\left(\frac{1}{2}-\Lambda,\nu\right),\left(\lambda+2-\frac{1}{2}\Lambda,\nu\right),(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}}\right]\tag{26}$$

4. Special Cases

1. If we Substitute $\tau_i = 1$ in (22) then Alphe (ℵ)-function reduces to I-function [7].
2. On taking $\tau_i = 1$, $r = 1$ and reducing $S_V^U[x]$ and $S^{\alpha', \beta', 0}[x]$ to unity in (22) the result reduces to a known result derived by Srivastava H.M. and Raina R.K. [5].
3. If we putting $\tau_i = 1$, $r = 1$, $A_j = B_j = 1$ and reducing $S_V^U[x]$ and $S^{\alpha', \beta', 0}[x]$ to unity in (25) the result reduces to another result known result derived by Srivastava H.M. and Raina R.K. [5].

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